A simplified approach for flexural behavior of epoxy resin materials

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Abstract
A piecewise-linear parametric uniaxial tension and compression stress–strain model with a simplified post-peak response is developed to obtain the nonlinear load deflection response of epoxy resin materials which are considerably stronger in compression than tension. This model could be used to obtain flexural strength when the complete post-peak behavior of the material in tension and compression is not available. The tension and compression stress–strain curves are bilinear for pre-peak response followed by a constant flow stress in tension and a constant yield stress in compression in the post-peak response. The simulations and experiments reveal the suitability of this model for predicting the three-point bending and four-point bending response. This model gives an upper bound estimate for flexural over-strength factor.

Keywords
Epoxy resin, stress–strain relations, moment distribution, curvature, load, deflection, three-point bending (3PB), four-point bending (4PB), nonlinear response

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Introduction
Epoxy resins are one of the most common matrix materials in fiber reinforced composites. However, their mechanical properties and the progressive failure patterns present a challenge for researchers investigating the integrity of these materials under different loading conditions. In structural applications, large flexural loads are considered one of many critical loading cases considered in the design of polymer-based composite structures. Developing a constitutive stress–strain relationship is difficult due to the need for characterization of mechanical behavior at different loading conditions. To the best of this authors’ knowledge, no analytical material characterization study exists in the literature that relates to the tension, compression, and flexural behavior of a single type of epoxy resin material.

Hydrostatic stresses are known to affect the yield stress and nonlinear response of epoxy resin materials.1 Several three-dimensional viscoelastic and/or viscoplastic constitutive models have been proposed for in-plane tension and compression material behavior.2–10 These models have been successful, especially, in fitting quasi-static test results. They have also been able to partially describe the material response at different strain rates. Jordan et al.11 and Lu et al.12 modified the constitutive models developed by Boyce et al.5 and Hasan and Boyce6 to obtain uniaxial compressive stress–strain behavior of epoxy resins. Bodner and Partom13 utilized a set of viscoplastic state variable constitutive equations to represent the inelastic behavior of Rene 95 at 650°C. Zhang and Moore14 and Gilat et al.15 used the Bodner–Partom theory13 to obtain the inelastic uniaxial response of polymers. Li and Pan,16 Chang and Pan,17 and Hsu et al.18 developed a model for polymeric materials by modifying the pressure-dependent Drucker–Prager yield criteria, originally introduced to deal with the plastic deformation of soils.19 A piecewise-linear tension and compression stress–strain relationship was used to study the mechanical behavior of high-performance fiber-reinforced cement composites by Naaman and Reinhardt.20 The piecewise-linear approach was used to study flexural behavior of cement-based composite materials.21–23 Three-point bending (3PB) tests were used to study the different environmental and aging effects on mechanical...
properties of different types of fiber posts. Digital speckle photogrammetry technique was used to study the effect of defects on flexural behavior of sandwich composite structures by Fergusson et al. Yekani Fard et al. studied the nonlinear mechanical behavior of Epon E 863 using the digital image correlation (DIC) system. Hobbiebrunken et al., Bazant and Chen, Odom and Adams, and Goodier studied the dependency of the failure and strength on the size effect, stress state, and volume of the body subjected to stress in epoxy resin polymers. Giannotti et al. and Vallo used the statistical Weibull analysis approach and estimated the mean flexural strength to be 40% higher than the tensile strength for a Weibull modulus greater than 14. Yekani Fard et al. used an analytical approach to evaluate the flexural over-strength factor in epoxy resin E 863. They observed that the flexural strength in 3PB beams with groove was at least 14% higher than the tensile peak stress at low strain rates. Flexural over-strength factor is the ratio of the flexural strength (peak stress) to the ultimate tensile strength (UTS). Figure 1 shows the concept of the flexural over-strength factor due to the stress gradient effect along a potential fracture path. Using flexural over-strength factor, the nominal uniaxial material capacities are increased.

Jordan et al., Yekani Fard et al., G’Sell and Souahi, Boyce and Arruda, Buckley and Harding, Shah Khan et al., Littell et al., Chen et al. and Gerlach et al. studied the tension and compression stress–strain curves in different polymers at different strain rates and environmental conditions. Epoxy resin materials exhibit the following distinct behavior in the tension and compression stress–strain behavior: linearly elastic, nonlinearly ascending pre-peak response, yield-like (peak) behavior, strain softening, and plastic flow or strain stiffening at high strain in some cases. Yekani Fard et al. showed that many common strain and deformation techniques such as strain gages, extensometers and actuators result either in no information of post-peak behavior or in an average strain over a specimen in the post-peak response. Averaging the strain over a specimen that is deforming non-homogeneously, especially in the plastic range, does not capture the post peak response accurately. Therefore, sufficient data of the post-peak mechanical behavior in tension and compression are not available.

In this study, a piecewise-linear tension and compression stress–strain model with constant plastic flow stress in tension and constant yield stress in compression in the post-peak response is used to obtain the flexural load deflection response in epoxy resin E 863, which is considerably stronger in compression than in tension. An inverse analysis technique is used to obtain the modified peak uniaxial strengths for the constant plastic stress model considering the flexural over-strength factor. The purpose of this study is two-fold: (i) to introduce a simplified modeling technique to obtain flexural strength of epoxy resins when sufficient data of post-peak behavior in tension and compression are not available; (ii) to compare the flexural over-strength factor for 3PB with groove and four-point bending (4PB) beams.

**Constant plastic stress for tension and compression**

To study the flexural response in E 863, Yekani Fard et al. used a piecewise-linear tension and compression stress and strain model with strain softening, curve fitted to the experimentally obtained, uniaxial stress–strain curves, as shown in Figure 2. While the compressive and tensile moduli are approximately equal for E 863, the stress of the first point showing deviation from linearity in the stress–strain curve and the peak stress in tension are lower than those in compression. This is the main reason that E 863 does not experience compression plastic flow in bending and their stress–strain

![Figure 1. Stress gradient effect on maximum flexural strength.](image1)

![Figure 2. Experiment and strain softening tension and compression model at 493 μstrain/sec.](image2)
relationship in the compression side is always in the ascending region and the first part of the softening regime, as shown by solid circles in Figure 3.

Therefore, a simplified tension and compression stress–strain model with constant plastic stress (Figure 4), might be more useful for modeling bending in epoxy resins, which are considerably stronger in compression than in tension. The proposed model consists of a bilinear ascending curve for both tension and compression in the pre-peak region. In the post-peak response, the strain softening is replaced by a constant yield stress in compression and a constant plastic stress in tension.

The two parameters characterizing the tensile response in the pre-peak region are proportionality elastic limit (PEL) and ultimate tensile strength (UTS). The post-peak region in the tension model is expressed with constant plastic stress ($\sigma_f$) and ultimate strain ($\varepsilon_{Ut}$). Yield stress is often assumed to be equal to the first peak stress in the stress–strain curve. The pre-peak region in compression is characterized by proportionality elastic limit in compression (PEL,c) and compressive yield stress (CYS). The post-peak response in compression is determined by yield stress and compressive ultimate strain ($\varepsilon_{Uc}$).

The strain model consists of two parameters: $E$, $\varepsilon_{PEL}$, $\mu_{t1}$, $\mu_{Ut}$, $\alpha$, $\omega$, $\gamma$, $\beta$, $\mu_{c1}$, and $\mu_{Uc}$. The tensile stress at the PEL point is related empirically to the stress at the UTS point. The ascending part of the tension and compression stress–strain curves consists of two linear parts: (a) 0 to PEL and PEL to UTS in tension, and (b) 0 to PEL and PEL to CYS in compression. The curve in post-peak response is idealized as horizontal, with $\sigma_f$ as the post-peak sustained stress in tension and $\sigma_{CYS}$ constant yield strength in compression. The constant normalized post-peak tensile stress level, $\omega$, exhibits the ability of the model to represent different levels of softening response. The post-peak response in tension and compression terminates at the ultimate tension strain level ($\varepsilon_{Ut} = \mu_{Ut} \varepsilon_{PEL}$) and ultimate compression strain level ($\varepsilon_{Uc} = \mu_{Uc} \varepsilon_{PEL}$), respectively. In the elastic range, elastic modulus in tension and compression for epoxy resin materials are practically identical. However, the material model could be treated as a bi-modulus in tension and compression. The nine normalized parameters used in the definition of the constitutive stress–strain curves are defined by
\[ \mu_{c0} = \frac{\varepsilon_{c0}}{\varepsilon_{PEL}}, \quad \mu_{c1} = \frac{\varepsilon_{c1}}{\varepsilon_{PEL}}, \quad \mu_{Uc} = \frac{\varepsilon_{Uc}}{\varepsilon_{PEL}} \]

\[ \mu_{t1} = \frac{\varepsilon_{t1}}{\varepsilon_{PEL}}, \quad \mu_{Ut} = \frac{\varepsilon_{Ut}}{\varepsilon_{PEL}} \]

\[ \gamma = \frac{E_c}{E}, \quad \beta = \frac{E_{PEL,c}}{E}, \quad \alpha = \frac{E_{PEL,t}}{E} \]

\[ \omega = \frac{\sigma_f}{E_{PEL}} \]

where \( \mu_{c0}, \mu_{c1}, \mu_{Uc} \) are normalized strain at the proportionality elastic limit point in compression, normalized strain at the CYS point, and normalized compressive strain at failure point, respectively. \( \mu_{t1}, \mu_{Ut} \) are normalized strain at UTS point and normalized tensile strain at the failure point. Stiffness parameters \( \alpha, \gamma, \beta \) are normalized stiffness at post PEL in tension, elastic stiffness in compression, and normalized post-PEL stiffness in compression. \( \omega \) is the normalized constant tensile softening stress.

**Moment curvature**

Figure 4 shows that there are three distinct regions on stress–strain curve for each tension and compression stress–strain relationship; therefore, there would be nine different cases of stress distribution across any arbitrary cross section, as shown in Figures 5 to 7. Since epoxy resin materials have higher mechanical strengths in compression than in tension, some cases of stress distribution are unlikely to occur for epoxy resin materials. However, an algorithm has been developed to account for all possible cases, so that any type of material exhibiting uniaxial tension and compression stress–strain as shown in Figure 4, could be modeled. Linear strain compatibility has been assumed in all these cases. Load is applied from 0 to failure by imposing a normalized compressive strain \( \varepsilon_{PEL} \) at top fiber. The area under the stress curves of each tension and compression subzone represents the internal longitudinal normal force. The tension and compression forces normalized to tension force at the PEL point \( (bhE_{c,PEL}) \) are summarized in Table 2. The internal normal forces at each step in time are balanced with the external applied moment.

Stress across a section develops from 0 to inelastic nonlinear in both tension and compression in case nine, shown in Figure 8, as the load is applied incrementally.

### Table 1. Definition of compression and tension stresses.

<table>
<thead>
<tr>
<th>Stress</th>
<th>Definition</th>
<th>Domain of strain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_c(\varepsilon_c) )</td>
<td>( E_c )</td>
<td>( 0 \leq \varepsilon_c \leq \varepsilon_{PEL} )</td>
</tr>
<tr>
<td>( E(\varepsilon_{PEL} + \alpha (\varepsilon_c - \varepsilon_{PEL})) )</td>
<td>( \omega E \varepsilon_{PEL} )</td>
<td>( \varepsilon_{PEL} &lt; \varepsilon_c \leq \mu_{c0} \varepsilon_{PEL} )</td>
</tr>
<tr>
<td>( \varepsilon_{c0} )</td>
<td>( \gamma E \varepsilon_c )</td>
<td>( \mu_{c0} \varepsilon_{PEL} &lt; \varepsilon_c \leq \mu_{c1} \varepsilon_{PEL} )</td>
</tr>
<tr>
<td>( E(\gamma \mu_{c0} \varepsilon_{PEL} + \beta (\varepsilon_c - \mu_{c0} \varepsilon_{PEL})) )</td>
<td>( E \varepsilon_{PEL} (\beta \mu_{c1} + \mu_{c0} (\gamma \beta)) )</td>
<td>( \mu_{c1} \varepsilon_{PEL} &lt; \varepsilon_c \leq \mu_{Uc} \varepsilon_{PEL} )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( \mu_{Uc} \varepsilon_{PEL} &lt; \varepsilon_c \leq \varepsilon_{Uc} \varepsilon_{PEL} )</td>
</tr>
</tbody>
</table>

**Figure 5.** (a) Rectangular cross section; (b) case 1: linear in compression and tension; (c) case 2: elastic in compression and post PEL in tension; and (d) case 3: post PEL in compression and elastic in tension.
at the top fiber. Stress may not develop up until stage six (Figure 8), considering the value of tension and compression parameters, but stress develops at least to stage four, where compressive failure is possible if $\lambda_{max} = \mu_{uc}$ in case 6, or tensile failure may happen when $\lambda_{max} = W$ in case four.

Stress evolution through the stages, shown in Figure 8, depends on the characteristic points (R to Z), which are functions of material parameters, and the controlling value $\lambda_{max}$. Transition points, defined by the parameter $tp_{ij}$ between different stages in Figure 8, are described as follows

\[
\begin{align*}
    tp_{12} &= \min(\mu_{c_0}, R) \\
    tp_{23} &= \min(\mu_{c_0}, T) \text{ or } \min(\mu_{c_1}, S) \\
    tp_{34} &= \min(\mu_{uc}, U) \text{ or } \min(\mu_{c_1}, V) \text{ or } \min(\mu_{c_0}, W) \\
    tp_{45} &= \min(\mu_{uc}, X) \text{ or } \min(\mu_{c_1}, Y) \\
    tp_{56} &= \min(\mu_{uc}, Z)
\end{align*}
\]

where indices $i$ and $j$ refer to origin and destination stages, respectively. Characteristic points R to Z are calculated as functions of material parameters to satisfy the following relation at each load step.

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Figure 6. (a) Case 4: elastic in compression and constant softening in tension; (b) case 5: post PEL in compression and tension; (c) case 6: constant yield in compression and elastic in tension; and (d) case 7: post PEL in compression and constant softening in tension.

Figure 7. (a) Case 8: constant yield in compression and post PEL in tension; and (b) case 9: constant yield in compression and constant softening in tension.
et is the tensile strain at the bottom fiber and V is one of the following \{1, m, ut\}. et is expressed as a linear function of the applied compressive strain at the top fiber (\(e_c\)), which is equal to \(l e_{PEL}\), and \(k\) is the depth of the neutral axis and is a function of material parameters

\[
\frac{\lambda}{\kappa} = l + \lambda + \frac{2k}{2}\kappa
\]

where \(\kappa\) is the tensile strain at the bottom fiber and \(\Omega\) is one of the following \{1, \(\mu_{t1}\), \(\mu_{c1}\)\}. \(et\) is expressed as a linear function of the applied compressive strain at the top fiber (\(e_c\)), which is equal to \(l e_{PEL}\), and \(k\) is the depth of the neutral axis and is a function of material parameters

\[
\frac{\lambda}{\kappa} = l + \lambda + \frac{2k}{2}\kappa
\]

As the applied load \(\lambda e_{PEL}\) is incrementally imposed, the strain and stress distribution is determined and the internal tension and compression forces are computed. The location of the neutral axis \(k\) throughout the

Table 2. Tension and compression forces for each subzone in all cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>(F_{t1}) or (F_{t1})</th>
<th>(F_{o}) or (F_{o})</th>
<th>(F_{o}) or (F_{o})</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(bh E e_{PEL})</td>
<td>(bh E e_{PEL})</td>
<td>(bh E e_{PEL})</td>
</tr>
<tr>
<td>1,3,6 tension</td>
<td>(\frac{\lambda}{k} - \lambda + \frac{2k}{2}\kappa)</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>2,5,8 tension</td>
<td>(\frac{\lambda}{2}\kappa) + (\frac{\lambda}{2}\kappa) + (\frac{\lambda}{2}\kappa) + (\frac{\lambda}{2}\kappa)</td>
<td>(-\frac{\lambda}{2\kappa}) + (\frac{\lambda}{2\kappa}) + (\frac{\lambda}{2\kappa}) + (\frac{\lambda}{2\kappa})</td>
<td>--</td>
</tr>
<tr>
<td>4,7,9 tension</td>
<td>(\frac{\lambda}{2}\kappa) + (\frac{\lambda}{2}\kappa) + (\frac{\lambda}{2}\kappa) + (\frac{\lambda}{2}\kappa)</td>
<td>(-\frac{\lambda}{2\kappa}) + (\frac{\lambda}{2\kappa}) + (\frac{\lambda}{2\kappa}) + (\frac{\lambda}{2\kappa})</td>
<td>--</td>
</tr>
<tr>
<td>1,2,4 compression</td>
<td>(\frac{\gamma}{2}\lambda\kappa)</td>
<td>(\frac{\gamma}{2}\lambda\kappa)</td>
<td>(\frac{\gamma}{2}\lambda\kappa)</td>
</tr>
<tr>
<td>3,5,7 compression</td>
<td>(\frac{\gamma}{2}\mu_{t1}\kappa)</td>
<td>(\frac{\gamma}{2}\mu_{t1}\kappa)</td>
<td>(\frac{\gamma}{2}\mu_{t1}\kappa)</td>
</tr>
<tr>
<td>6,8,9 compression</td>
<td>(\frac{\gamma}{2}\mu_{t1}\kappa)</td>
<td>(\frac{\gamma}{2}\mu_{t1}\kappa)</td>
<td>(\frac{\gamma}{2}\mu_{t1}\kappa)</td>
</tr>
</tbody>
</table>

Figure 8. Stress development at different stages of flexural loading.

\[ e_t = \Omega e_{PEL} \]

\[ e_t = \frac{1 - \kappa}{\kappa} e_c \]
loading is obtained by imposing the equilibrium condition at each case for each step of load

\[
\sum_{i=1}^{3} F_{ei} - \sum_{j=1}^{3} F_{dj} = 0 \Rightarrow \kappa_m(\alpha, \beta, \gamma, \omega, \mu_{c1}, \mu_{U1}, \mu_{U2}, \lambda)
\]

(7)

where \( F_{ei} \) and \( F_{dj} \) for \((i = 1, 2, 3)\) are tension and compression forces calculated from the stress diagrams. Tension and compression forces for case nine are presented in Appendix 2. The equilibrium governing equation in some cases results in more than one solution for \( \kappa \), but the valid value of \( \kappa \) is between 0 and 1. Figure 9 shows the negligible unbalanced normalized internal force using the correct expressions for \( \kappa \) throughout loading. For a beam with a cross section of \( 4 \times 7 \text{ mm}^2 \), \( E = 3049 \text{ MPa} \), and \( \phi_{PEL} = 0.0162 \), the amount of unbalanced internal force is \( 1.4 \times 10^{-16} \text{ N} \), which is negligible. The expressions for \( \kappa \) are presented in Appendix 3. By substituting the expressions for \( \kappa \) in equation (5), the characteristic points \( R \) to \( Z \) as functions of material parameters are calculated.

As an example, the steps to obtain the normalized moment and curvature expressions for case nine in Figure 7 are explained in detail in Appendix 2. The general definitions for normalized moment and curvature are shown in equations (8) through (10)

\[
M(\lambda, \gamma, \beta, \alpha, \mu_{c0}, \mu_{c1}, \mu_{U1}, \mu_{U2}, \omega) = M_{PEL} M' \kappa_m(\alpha, \beta, \gamma, \omega, \mu_{c0}, \mu_{c1}, \mu_{U1}, \mu_{U2}, \mu_{U3}, \omega)
\]

(8)

\[
\varphi(\lambda, \gamma, \beta, \alpha, \mu_{c0}, \mu_{c1}, \mu_{U1}, \mu_{U2}, \omega) = \varphi_{PEL} \kappa_m(\alpha, \beta, \gamma, \omega, \mu_{c0}, \mu_{c1}, \mu_{U1}, \mu_{U2}, \mu_{U3}, \omega)
\]

(9)

\[
\varphi'(\lambda, \gamma, \beta, \alpha, \mu_{c0}, \mu_{c1}, \mu_{U1}, \mu_{U2}, \omega) = \frac{\lambda}{2 \kappa_m}, \quad i = 1, 2, 3, \ldots, 9
\]

(10)

where \( M_{PEL} \) and \( \varphi_{PEL} \) are moment and curvature at the elastic limit, and are defined as follows.

\[
M_{PEL} = \frac{bh^2 E_{PEL}}{6}, \quad \varphi_{PEL} = \frac{2\phi_{PEL}}{h}
\]

(11)

The expressions for the normalized moment are summarized in Appendix 3. If there is no intrinsic flaw in a material, \( M_u \) could approach \( M_\infty \) for very large \( \lambda \) values. In this ideal situation, the normalized moment at very large \( \lambda \) values, \( M_\infty \), is computed by substituting \( \lambda = \infty \) in the expression for \( \kappa \) in case nine, and by substituting \( \lambda = \infty \) and \( \kappa_\infty \) in the normalized moment expression. Equations (12) through (14) present the values of the neutral axis depth, normalized moment, and curvature for very large \( \lambda \) values

\[
\kappa_\infty = \frac{\omega}{\omega + \gamma \mu_{c0} - \beta \mu_{c0} + \beta \mu_{c1}}
\]

(12)

\[
M_\infty = \frac{3\omega(\gamma \mu_{c0} - \beta \mu_{c0} + \beta \mu_{c1})}{\omega + \gamma \mu_{c0} - \beta \mu_{c0} + \beta \mu_{c1}}
\]

(13)

\[
\phi_\infty = \infty
\]

(14)

The neutral axis depth and normalized moment are functions of material parameters in compression (\( \gamma, \beta, \mu_{c0}, \mu_{c1} \)) and constant normalized tensile plastic stress (\( \omega \)). Expression \( \gamma \mu_{c0} + \beta \) is the definition of normalized \( \sigma_{CYS} \). For flawless epoxy resin materials with very low tensile plastic stress, the normalized moment is almost three times the normalized tensile plastic stress. The normalized neutral axis depth ratio and moment for an elastic perfectly plastic material as logically expected are 0.5 and 1.5, respectively.44 For a set of parameters, \( \gamma, \beta, \mu_{c0}, \mu_{c1} \), the critical value of \( \omega \), which results in a flexural capacity at infinity (failure) greater than the flexural capacity at the tensile PEL point, can be found. By equating the normalized moment for large top compressive strain to one (\( M_\infty = 1 \)), the critical value of normalized tensile plastic stress, \( \omega_{critical} \), is

\[
\omega_{critical} = \frac{\gamma \mu_{c0} - \beta \mu_{c0} + \beta \mu_{c1}}{3(\gamma \mu_{c0} - \beta \mu_{c0} + \beta \mu_{c1}) - 1}
\]

(15)

The required parameters for Epon E 862\textsuperscript{40} and E 863\textsuperscript{27} have been defined through curve fitting to the tension and compression stress–strain curves. Calculations show the \( \omega_{critical} \) for E 862 and E 863 as 0.39 and 0.4, respectively (almost 25% of the ultimate tensile strength).

**Parametric study**

Yekani Fard et al.\textsuperscript{34,35} studied the effect of different parts of strain softening tension and compression stress–strain models on the flexural response. A set of parametric studies based on the proposed constant plastic stress model were performed to determine the important parameters for flexural load-carrying capacity. Tension and compression stress–strain curves of Epon E 862\textsuperscript{40} was chosen as the base parameters: \( E = 2069 \text{ MPa} \),
MPa, \( E = 2457 \) MPa, \( \varepsilon_{PEL} = 0.0205 \), \( \varepsilon_{Uts} = 0.076 \), 
\( \varepsilon_{UTS} = 0.24 \), \( \varepsilon_{PEL,U} = 0.019 \), \( \varepsilon_{CYS} = 0.092 \), \( \varepsilon_{Uc} = 0.35 \), 
\( \sigma_{UTS} = 70 \) MPa, \( \sigma_f = 60.5 \) MPa, \( \sigma_{CYS} = 93 \) MPa.

Figures 10(a) and (b) depict the effect of ultimate tensile stress with constant PEL slope and constant tensile softening stress on the moment curvature and neutral axis location. Figure 10(b) reveals that an increase in \( \mu_{\alpha} \) considerably increases flexural strength. However, the amount of moment at failure is not affected as much as the flexural strength, since for \( \mu_{\alpha} > 6 \), the moment at infinity is less than the flexural strength. The effectiveness of ultimate tensile strength on moment-carrying capacity was also observed using the softening model developed by Yekani Fard et al.\(^{34}\) It is possible to look at the variation of \( \mu_{\alpha} \) as the variation of \( \sigma_{CYS} \) to \( \sigma_{UTS} \) ratio defined as

\[
\frac{\gamma \mu_{\alpha} + \beta (\mu_{\alpha} - \mu_{\alpha0})}{1 + \alpha (\mu_{\alpha} - 1)}
\]

By substituting \( \gamma = 1.19, \beta = 0.3, \alpha = 0.24, \mu_{\alpha0} = 0.93, \mu_{\alpha1} = 4.49 \), it is clear that changes in \( \mu_{\alpha} \) from 2.75 to 8 will change the \( \sigma_{CYS} \) to \( \sigma_{UTS} \) ratio from 1.53 to 0.81.

The effects of strain at constant ultimate tensile stress on the flexural behavior are presented in Figure 11(a) and (b). Figure 11 reveals that changes in parameters \( \alpha \) and \( \mu_{\alpha1} \) slightly affect the moment but extremely affect the position of the neutral axis for a wide range of normalized top compressive strains between one and four, which in turn will change the stress distribution across the cross section between elastic and post-peak range. For a resin material without any intrinsic flaws, the analysis indicates that tension is the governing failure mechanism in all cases, while compression strain just exceeds the yield point.

**Algorithm for load deflection response**

An experimental investigation of the tension, compression, and 3PB flexural behavior of Epon E 863 was presented by Yekani Fard et al.\(^{26,27}\) It was shown that Epon E 863 demonstrates strain softening behavior in tension and compression, and higher yield and softening stresses in compression than ultimate tensile and softening stresses in tension. Based on these observations, a strain softening tension and compression model was used to obtain the nonlinear moment curvature response for Epon E 863.\(^{34,35}\) It was observed that the strain softening model accurately captures the moment curvature response in pre-peak and post-peak portions. After curve fitting the constant softening stress model to the experimental tension and compression stress–strain curves, the normalized moment curvature response is obtained. Figure 12 compares the normalized moment curvature diagram from the proposed constant plastic flow model and the strain softening model\(^{34,35}\) at 493 \( \mu \)str/sec. The constant flow stress model differs slightly from the strain softening model in the pre-peak portion of the response. However, it cannot predict the softening behavior in the moment curvature response. This is mainly due to the perfectly plastic model in compression. The proposed constant plastic flow model shows the same moment carrying capacity for E 863 as the softening model does, but at different curvatures and only if there is no intrinsic flaw in the material. The load deflection response is obtained using the nonlinear moment curvature response and establishing static equilibrium. In displacement control, the normalized top compressive strain is incrementally imposed from 0 to failure, in order to generate a stress distribution profile in a given cross section.
For resins such as E 862 and E 863, since the compressive strength is greater than the tensile strength, the shape of the moment curvature diagram greatly depends on the tensile stress–strain model, as observed in the parametric study. The constant flow curve in Figure 12 shows a typical nonlinear moment curvature diagram, based on the constant plastic flow stress model for epoxy resins, consisting of a linear elastic part followed by an ascending curve with reduced stiffness.

The first deviation from linearity in a moment curvature or load deflection curve is called limit of proportionality (LOP). The specimen is loaded from 0 to $P_{LOP}$ in the linear portion of the moment curvature diagram from 0 to $M_{LOP}$. The curvature for the linear elastic portion is determined directly from the moment curvature diagram. Beyond the LOP, the curvature is obtained from the nonlinear portion of the moment curvature diagram. Static equilibrium is used to obtain a series of load steps in 3PB or 4PB setups from the moment curvature diagram. The main steps to calculate load deflection response are summarized as follows.

1. Calculate transition points to determine the possible cases of stress distribution based on material properties for a piecewise-linear model.
2. Impose load incrementally by increasing the normalized compressive strain in top fiber to obtain the nonlinear moment curvature response using closed-form expressions for moment and curvature relevant to the cases in step one.
3. Calculate applied load vector ($P = 2 M/S$ where $S$ is the span and the distance between a support and adjacent load for the 3PB and 4PB, respectively).
4. Calculate moment diagram along the structure for any load in step three.
5. Determine curvature diagram for any load in step three along the structure using moment curvature relationship.
6. Calculate amount of deflection using one of the methods for statically determinate structures (e.g. virtual work method or moment area method).
7. Repeat steps three to six for each load.

Experiments

Tension, compression, 3PB bending beams with groove, and 4PB beam tests were conducted on epoxy resin Epon E 863 with a hardener EPI-CURE 3290 using a
100/27 weight ratio at room temperature and at low loading speeds. The DIC system, ARAMIS 4M, was used to study the strain fields. Dog bone samples with a gage length of 14 mm and a rectangular cross-section of $3.18 \times 3.43$ mm$^2$ were selected to conduct the monotonic tensile tests. Small cubic samples (4 $\times$ 4 $\times$ 4 mm$^3$), right square-sided prisms with length of 8 mm and side of 3.5 mm and cylinders with a length of 10 mm and diameter of 4 mm were tested under monotonic compression. Small beams with the width of 4 mm, thickness of 10 mm, and length of 60 mm (50 mm span) were selected to conduct 3PB and 4PB tests. A groove (radius of approximately 3.5 mm) was cut in the middle of the beam for 3PB tests. The middle span in the 4PB tests was 25 mm. Details of the experiments were explained in the authors’ other publications.

Simulation of 3PB and 4PB flexural load deflection response

The strain softening model for both tension and compression, developed by Yekani Fard et al., demonstrated that the uniaxial tension and compression stress–strain curves underestimate the load deflection response due to a difference between the stress distribution profile in the uniaxial and bending tests. Figure 13 illustrates the representative experimental tension and compression true stress–strain curves at 493 $\mu$str/sec. As stated before, Epon E 863 has a strain softening behavior in tension and compression. The constant plastic stress model was built through curve fitting and is shown as a solid line in Figure 13. The two main parameters and nine non-dimensional parameters for the model at 493 $\mu$str/sec are: $E = 3049$ MPa, $\varepsilon_{PEL} = 0.0162$, $\mu_{c0} = 1.148$, $\mu_{c1} = 3.52$, $\mu_{c2} = 6.79$, $\mu_{Uc} = 15.70$, $\mu_{c1} = 2.55$, $\mu_{Uc} = 20.98$, $\gamma = 1.09$, $\alpha = 0.395$, $\beta = 0.298$, and $\omega = 1.369$. Simulations were conducted to study the flexural load deflection response of Epon E 863 and to evaluate the suitability of the constant flow stress model and the effects of out-of-plane loading. Figure 14 shows the 3PB and 4PB load deflection curve compared with the simulation results. This figure illustrates that the direct use of tension and compression stress–strain model follows the flexural response in the pre-peak portion and underestimates the flexural strength as it is logically expected. However, the simulated load deflection curve obtained from the constant flow stress model is not able to capture the deflection softening behavior. The main reason for the underestimation of the flexural strength is that, in tension and compression, the entire volume of the sample is subjected to the same load as shown in Figure 1 and has the same probability of failure. Therefore, the probability of crack nucleation, propagation, and failure development in tension and compression samples is higher than in bending samples. Results of the parametric study show that flexural load-carrying capacity can be improved by increasing the ultimate tensile and compressive levels through an over-strength flexural factor (parameter $C_1$) and by further adjustments to the other parameters. The parameter $C_1$ was calculated to modify the uniaxial strain softening tension and compression stress–strain model for flexural simulation. Figure 14 shows underestimation in the prediction of load carrying capacity for 3PB and 4PB using constant flow stress model. In order to quantify parameter $C_1$, an inverse analysis technique has been used to obtain the stress–strain model.

An inverse analysis of the 3PB and 4PB load deflection response showed that the value of $C_1$ for Epon E
863 was around 1.30 and 1.55, respectively. The reason for the higher factor in 4PB than 3PB is that in 3PB, with a groove or notch at the center, the beam is forced to fracture at the location of the groove or notch caused by stress concentration. However, the beam’s center may not be the weakest point in the beam, so the groove and notch may influence the actual strength of the material. The results from the analytical approach could be compared with the results from Weibull analysis approach\(^{32,33}\) which estimated the mean flexural strength to be 40% higher than the tensile strength. More studies at different strain rates and on different epoxy resin materials need to be performed before an average flexural over-strength factor can be recommended.

**Conclusions**

A piecewise-linear stress–strain relation with simplified post-peak response in tension and compression for flexural simulation of epoxy resin materials has been proposed. This model consists of a bilinear ascending curve in the pre-peak portion and constant flow stress in the post-peak tension and compression. The material model is described using two intrinsic material parameters (tensile modulus of elasticity and tensile strain at the PEL point), in addition to nine non-dimensional parameters for tension and compression. It was concluded that compression stress–strain parameters have less effect on flexural behavior than tension parameters, as long as material is stronger in compression than in tension. Epoxy resin materials with a considerable amount of post-peak tensile strength have a moment capacity of approximately 2.4 times the elastic moment. Simulation of the load deflection response of epoxy resins in 3PB and 4PB test revealed the effect of stress gradient on material behavior. Results indicate that direct use of tension and compression data underestimates the flexural strength. The prediction of flexural load carrying capacity can be improved by applying a scaling factor (\(C_1\)) to uniaxial tension and compression strength. However, the value of \(C_1\) is higher for 4PB than 3PB due to the effect of stress concentration at the location of groove in the 3PB beam.

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**References**


45. ARAMIS. *User’s manual for 3-D image photogrammetry,* 2006.
Appendix 1

Notation

- $b$: beam width
- $E$: modulus of elasticity in tension (if $\gamma \neq 1$) or modulus of elasticity (if $\gamma = 1$)
- $E_c (\gamma)$: modulus of elasticity in compression if $\gamma \neq 1$ (normalized term respect to $E$)
- $E_{PEL-c}$: stiffness at the post proportionality limit in compression (normalized term respect to $E$)
- $E_{PEL-t}$: stiffness at the post proportionality limit in tension (normalized term respect to $E$)
- $F_i$: force component in each sub-zone ($i = 1, 2, 3$) of stress diagram
- $h$: beam depth
- $M_j$: moment for each case of stress
- $(M'_j)$: distribution across the depth (normalized moment)
- $R$ to $Z$: characteristic points
- $e_c$: compressive strain
- $e_{cYS}$: strain at the compressive yield strength
- $(\mu_{c})$: (peak) point (normalized term respect to $E_{PEL}$)
- $e_{PEL}$: strain at the proportionality elastic limit point in tension
- $e_{PEL,c}$: strain at the proportionality elastic limit point in compression (normalized respect to $E_{PEL}$)
- $e_t$: tensile strain
- $e_{Uc}$: strain at the compressive failure point
- $(\mu_{Uc})$: (normalized term respect to $E_{PEL}$)
- $e_{Uts}$: strain at ultimate tensile strength (peak)
- $(\mu_{Uts})$: point (normalized term respect to $E_{PEL}$)
- $e_{Ut}$: strain at ultimate tensile strength (peak)
- $(\mu_{Ut})$: point (normalized term respect to $E_{PEL}$)
- $\kappa$: neutral axis depth ratio for each case of stress distribution
- $\lambda$: normalized applied top compressive strain ($e_{c/EPEL}$)
- $\omega$: normalized tensile post-peak stress respect to tensile elastic stress
- $\Omega$: auxiliary parameter which is one of the $\{1, \mu_{11}, \mu_{12}\}$
- $\varphi (\varphi^{'})$: curvature (normalized curvature)

Appendix 2

Calculation of normalized moment and curvature for case 9

$$F_{C91} = \frac{\gamma E b k h (\mu_{c1} - \mu_{c0})(2\gamma \mu_{c0} + \beta (\mu_{c1} - \mu_{c0}))}{2\lambda} \quad (16)$$

$$F_{C92} = \frac{E b k h (\mu_{c1} - \mu_{c0})(2\gamma \mu_{c0} + \beta (\mu_{c1} - \mu_{c0}))}{2\lambda} \quad (17)$$

$$F_{C93} = \frac{E b k h (1 - \frac{\mu_{c1}}{\lambda}) (\gamma \mu_{c0} + \beta (\mu_{c1} - \mu_{c0}))}{E_{PEL}} \quad (18)$$

$$F_{t01} = \frac{E b k h}{2\lambda} \quad (19)$$

$$F_{t02} = \frac{E b k h (\mu_{c1} - 1)(2 + \alpha (\mu_{c1} - 1))}{2\lambda} \quad (20)$$

$$F_{t03} = E b h \sigma \left(1 - \kappa - \frac{\kappa \mu_{c1}}{\lambda}\right) \quad (21)$$

$$Z_{c91} = \frac{2 \mu_{c0} k h}{3 \lambda} \quad (22)$$

$$Z_{c92} = \frac{\mu_{c0} \kappa h}{\lambda} + \frac{\kappa h (\mu_{c1} - \mu_{c0}) (3 \gamma \mu_{c0} + 2\beta (\mu_{c1} - \mu_{c0}))}{3\lambda (2\gamma \mu_{c0} + \beta (\mu_{c1} - \mu_{c0}))} \quad (23)$$

$$Z_{c93} = \frac{\mu_{c1} \kappa h}{\lambda} + \frac{1}{2} k h \left(1 - \frac{\mu_{c1}}{\lambda}\right) \quad (24)$$

$$Z_{t01} = \frac{2 \kappa h}{3 \lambda} \quad (25)$$

$$Z_{t02} = \frac{\kappa h}{\lambda} + \frac{k h (\mu_{c1} - 1)(3 + 2\alpha (\mu_{c1} - 1))}{6 + 3\alpha (\mu_{c1} - 1)} \quad (26)$$

$$Z_{t03} = \frac{\mu_{c1} \kappa h}{\lambda} + \frac{1}{2} h \left(1 - \kappa - \frac{\kappa \mu_{c1}}{\lambda}\right) \quad (27)$$

$$\sum_{i=1}^{3} F_{i} - \sum_{j=1}^{3} F_{ij} = 0 \Rightarrow \kappa_{9}$$

$$= \frac{2\omega}{(1 - 2\mu_{11} - \alpha (\mu_{11}^2 + 1) + 2\omega (\lambda + \mu_{11}) + 2\alpha \mu_{11}}$$

$$+ \beta \mu_{11}^2 + (\beta - \gamma) \mu_{c0}^2 + 2\gamma \lambda \mu_{c0} + 2\beta \lambda (\mu_{c1} - \mu_{c0})) \quad (28)$$

$$mm_{90} = \frac{6}{b k h^2 E_{PEL}} \left(\sum_{i=1}^{3} F_{C9i} Z_{C9i} + \sum_{i=1}^{3} F_{t0i} Z_{t0i}\right)$$

$$mm_{90} = \frac{k^2}{\lambda^2} \left(\frac{\mu_{c0}^3 (\beta - \gamma) + 3\mu_{c0} \lambda^2 (\gamma - \beta) - 1}{3\mu_{11}^2 (1 - \alpha) + \alpha + 3\beta \mu_{c1} \lambda^2 + 3\alpha (\lambda^2 - \mu_{c1}^2)} \right)$$

$$+ \frac{k^2}{\lambda^2} \left(2\alpha \mu_{c1}^3 - \beta \mu_{c1}^3\right) - 6\omega \kappa + 3\omega \quad (29)$$

$$\varphi_{3} = \frac{h \lambda_{PEL}}{2\omega_{PEL} \kappa_{9} h} = \frac{\lambda}{2\kappa_{9}} \quad (30)$$
### Appendix 3 Neutral axis depth ratio and normalized moment

<table>
<thead>
<tr>
<th>Case</th>
<th>( \kappa_i )</th>
<th>( M'_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(-1 + \sqrt{y} )</td>
<td>( 2\lambda (y - 1)\kappa_i^2 + 6\lambda \kappa_i - 6\lambda + \frac{2\lambda}{\kappa_i} )</td>
</tr>
<tr>
<td>2</td>
<td>(-\lambda (\alpha + 1) - 1 - \sqrt{-\alpha + 1 + \alpha \gamma \lambda^2} )</td>
<td>( \frac{t_i}{1 + \gamma \lambda^2} )</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{2\alpha \lambda}{t_4 + \gamma \lambda^2} )</td>
<td>( t_i = \frac{(\lambda - \mu_{i\text{e}})^2 (\beta - \gamma) + \gamma \lambda^2}{\kappa_i} \lambda )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{2\alpha \lambda}{t_4 + \gamma \lambda^2} )</td>
<td>( t_i = \frac{(\lambda - \mu_{i\text{e}})^2 (\beta - \gamma) + \gamma \lambda^2}{\kappa_i} \lambda )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{\lambda (\alpha + 1) - 1 - \sqrt{-\alpha + 1 + \alpha \gamma \lambda^2}}{t_1 - t_2} )</td>
<td>( t_i = \frac{(\lambda - \mu_{i\text{e}})^2 (\beta - \gamma) + \gamma \lambda^2}{\kappa_i} \lambda )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{\lambda (\alpha + 1) - 1 - \sqrt{-\alpha + 1 + \alpha \gamma \lambda^2}}{t_1 - t_2} )</td>
<td>( t_i = \frac{(\lambda - \mu_{i\text{e}})^2 (\beta - \gamma) + \gamma \lambda^2}{\kappa_i} \lambda )</td>
</tr>
<tr>
<td>7</td>
<td>( \frac{2\alpha \lambda}{t_4 + \gamma \lambda^2} )</td>
<td>( t_i = \frac{(\lambda - \mu_{i\text{e}})^2 (\beta - \gamma) + \gamma \lambda^2}{\kappa_i} \lambda )</td>
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<tr>
<td>8</td>
<td>( \frac{2\alpha \lambda}{t_4 + \gamma \lambda^2} )</td>
<td>( t_i = \frac{(\lambda - \mu_{i\text{e}})^2 (\beta - \gamma) + \gamma \lambda^2}{\kappa_i} \lambda )</td>
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<tr>
<td>9</td>
<td>( \frac{2\alpha \lambda}{t_4 + \gamma \lambda^2} )</td>
<td>( t_i = \frac{(\lambda - \mu_{i\text{e}})^2 (\beta - \gamma) + \gamma \lambda^2}{\kappa_i} \lambda )</td>
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